



Projectivity

Main Theorem: X s.m. curve, $g \geq 2$. Then $M_X^{ss}(r, d)$
& $M_X^{ss}(r, L)$ are proj. scheme of dimension
 $r^2(g-1)+1$ & $(r^2-1)(g-1)$ respectively.

Observation: injectivity of $M_X^{ss}(r, d)$ follows from that of $M_X^{ss}(r, L)$

$$\det : M_X^{ss}(r, d) \rightarrow \text{Pic}^d(X)$$

$$\det : \underbrace{M_X^{ss}(r, d)}_{\substack{\text{proj. alg. space} \\ \text{of}}} \longrightarrow \text{Pic}^d(X) \quad \text{inj.} \quad \checkmark$$

$$\det^{-1}([L]) = M_X^{ss}(r, L)$$

$\Rightarrow$ suffices to produce a rel. ample l.b. on M_X^{ss}

Claim: determinant l.b. L_V works. $\forall V$ satisfying (*)

Recall: V : v.b. on X . $\mathcal{L}_V := (\det R_{p_*}(\mathcal{L}^* V \otimes \mathcal{E}_{\text{inv}}))^{\vee}$

$$\mathcal{M}_X^{\text{ss}} \times X \xrightarrow{\pi} X$$

$$\uparrow \downarrow$$

$$\mathcal{M}_X^{\text{ss}}$$

$$\chi(V \otimes E)$$

v = Chern characters
= (r, d)

$$\text{If } \chi(V \otimes E) = 0, \forall E \in \mathcal{M}_X(r, d), \quad (*)$$

$$\text{then } \text{rk}(R_{p_*}(\mathcal{L}^* V \otimes \mathcal{E}_{\text{inv}})) = 0.$$

$$[K^0 \xrightarrow{\beta} K^1]$$

$$\det(R_{p_*}(\mathcal{L}^* V \otimes \mathcal{E}_{\text{inv}})) := \det(K^0) \otimes \det(K^1)^{\vee}$$

$$\exists \text{ a section } S_V = \det(\beta) \in \Gamma(\det(-)^{\vee})$$

Set-up: $h := \gcd(r, d)$. $r = hr_1$, $d = hd_1$. for some m

$$V: \text{v.b. on } X \text{ w/ } \text{rk} = mr_1, \text{ deg} = mr_1(g-1) - md_1$$

$$(\Rightarrow \chi(V \otimes E) = 0), \quad \forall E \in \mathcal{M}_X(r, d)$$

$$S_V(E) \neq 0 \Leftrightarrow V \otimes E \text{ is BN-gen'l.} \quad \text{cobordy} \quad := \text{at most one non-zero}$$

Lemma 5.7: $\{\mathcal{F}_\alpha\}$ bounded family of v.b. of $\text{rk} = r$ on X .

Then $\exists$ a scheme S of f. type $/k$ of $\dim \leq r^2(g-1) + 1$ & a v.b. $\mathcal{F}$ on $S \times X$ s.t. each $\mathcal{F}_\alpha$ appears as $\mathcal{F}_s \quad \exists s \in S$.

Lemma 5.8: E : v.b. $\text{rk} = r$, $\deg = d$. $\varepsilon \in \mathbb{Z}$. $m \gg 0$.

V : gen'l stable v.b. w/ $\text{rk} = mr_1$
 $\deg = mr_1(g-1) - md_1 + \varepsilon$

i) $\nexists V^\vee \rightarrow E$ w/ image F s.t. $\mu(F) < \mu(E)$

ii) if moreover E stable & $\varepsilon \leq 0$, $\nexists$ non-zero $V^\vee \rightarrow E$

Proof: As $m \gg 0$, may assume $\text{rk } V > r$.

Note: $\text{rk } V^\vee = mr_1$, $\deg V^\vee = mr_1(1-g) + md_1 - \varepsilon$

if $V^\vee \rightarrow E$, then $\mu(F) > \mu(V^\vee) = \mu(E) - g + 1 - \frac{\varepsilon}{mr_1}$
 $\Rightarrow \begin{cases} \mu(E) - g + 1 - \varepsilon, & \varepsilon \geq 0 \\ \mu(E) - g + 1, & \varepsilon \leq 0 \end{cases}$

As $\text{rk}(F) \leq r$, if $\mu(F) \leq \mu(E)$.

$\exists$ finitely many options for $\text{rk}(F)$ & $\deg F$.

Fix $1 \leq k \leq r$ & $l/k < \mu(E)$.

$$\begin{array}{ccccccc} 0 & \rightarrow & G & \rightarrow & V^\vee & \rightarrow & F \rightarrow 0 \\ & & & & & \searrow & \downarrow \\ & & & & & & E \end{array}$$

$$\text{rk } F = k, \quad \deg F = l$$

$$\text{rk } G = mr_1 - k, \quad \deg G = mr_1(1-g_1) + md_1 - e - l$$

$$\dim \mathcal{U}(F) \leq k^2(g-1) + 1$$

$$\dim \mathcal{U}(G) \leq (mr_1 - k)^2(g-1) + 1$$

$$\text{Hom}(F, G) = 0 \quad \text{ext}^1(F, G) \xrightarrow{\quad} -\chi(F^\vee \otimes G) = -\deg(F^\vee \otimes G) - \text{rk}(F^\vee \otimes G)(1-g_1)$$

$$\text{as } V^\vee \rightarrow F$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ V^\vee & \hookrightarrow & G \end{array}$$

$$= \deg(F) \text{rk}(G) - \text{rk}(F) \deg G + \text{rk } F \text{rk } G (g-1)$$

$$= l(mr_1 - k) - k(mr_1(1-g_1) + md_1 - e - l) + k(mr_1 - k)(g-1)$$

$$= \underbrace{(l r_1 - k d_1)}_{\text{"}} m + k r_1 (g-1) m - k^2 (g-1) + k e$$

$$k r_1 (\mu(F) - \mu(E)) < 0$$

$$\Rightarrow \dim \mathcal{U}(V') \leq \dim \mathcal{U}(F) + \dim \mathcal{U}(G) + \text{ext}^1(F, G) - 1$$

$$\leq (mr_1)^2(g-1) + kr_1(\underbrace{\mu(F) - \mu(E)}_{< 0 \text{ for } m \gg 0})m + k^2(g-1) + k\varepsilon + 1$$

$$< (mr_1)^2(g-1) + 1$$

$$= \dim M_x^s(mr_1, d_0)$$

$\Rightarrow$ gen'l V satisfies i)

ii) Using i) & E stable $\Rightarrow F = E$

$$0 \rightarrow G \rightarrow V' \rightarrow E \rightarrow 0$$

$$\dim \mathcal{U}(V') \leq \dim \mathcal{U}(G) + \text{ext}^1(E, G) - 1$$

$$= (mr_1)^2(g-1) + r\varepsilon$$

$$\stackrel{\varepsilon \leq 0}{<} (mr_1)^2(g+1) + 1$$

$$= \dim M_x^s(mr_1, d_0)$$

□

Lemma 5.9: $E_0, \dots, E_n$ stable v.b., $\mu = d/r$, $\text{rk } E_i = k_i$

Then for $m \gg 0$, a gen'l stable V w

$$\text{rk } V = mr_1 \quad \& \quad \deg V = mr_1(g-1) - md_1 + 1$$

satisfies:

i) $h^0(V \otimes E_i) = k_i \quad (\Leftrightarrow h^1(V \otimes E_i) = 0)$

ii) every non-zero $V^\vee \rightarrow E_i$ is surj. $\forall 0 \leq i \leq n$

iii) $\forall 1 \leq i \leq n$, $\forall$ non-zero $V^\vee \rightarrow E_i$ & $V^\vee \rightarrow E_0$,

$$\leadsto V^\vee \rightarrow E_0 \otimes E_i \text{ is surj.}$$

pf) $\text{hom}(V^\vee, E_i) = h^0(V \otimes E_i) = \chi(V \otimes E_i) + h^1(V \otimes E_i)$

$$\chi(V \otimes E_i) = \deg(V \otimes E_i) + \text{rk}(V \otimes E_i)(1-g)$$

$$= \deg V \cdot \text{rk } E_i + \deg E_i \cdot \text{rk}(V) + \text{rk } V \cdot \text{rk } E_i \cdot (1-g)$$

$$\deg E_i / k_i = \mu = \frac{d_1}{r_1}$$

$$= k_i(mr_1(g-1) - md_1 + 1) + mk_i d_1 + mr_1 \cdot k_i(1-g)$$

$$= k_i$$

$\Rightarrow$ i) Suffices: $\forall$ gen'l stable V , $h^1(V \otimes E_i) = 0 \quad \forall i$

$$h^1(V \otimes E_i) = h^1(V^\vee \otimes \omega_X \otimes E_i^\vee) = \dim(V \otimes \omega_X^\vee, E_i^\vee)$$

$\left. \begin{array}{l} \text{gen'l stable} \\ \text{stable, slope} = -d_1/r_1 \end{array} \right\}$
 $rk = mr_1, \deg = mr_1(g-1) + md_1 - 1$

$$\xrightarrow[\substack{\text{Lemma 5.8 (ii)} \\ \Sigma=1, E=E_i^\vee}]{\text{}} \text{}$$

$$\dim(V \otimes \omega_X^\vee, E_i^\vee) = 0 \quad \forall m \gg 0 \quad \text{gen'l } V.$$

i) ✓

May also assume that for gen'l stable V , the

image of any non-zero $V^\vee \rightarrow E_i$ has slope μ

(Lemma 5.8 i) $\Sigma=1$)

As E_i is stable, the map is surj. ii) ✓

$$\xrightarrow[\substack{\text{Lemma 5.8 i)} \\ E=E_0 \oplus E_i}]{\text{}} \text{}$$

$\forall$ gen'l stable V , $\forall$ non-zero $V^\vee \rightarrow E_0 \oplus E_i$

has image F w/ slope μ .

Since $V^\vee \rightarrow E_0$, $V^\vee \rightarrow E_i$, & any non-zero subsheaf

of $E_0 \oplus E_i$ w/ slope μ is $E_0, E_i, E_0 \oplus E_i$

$\Rightarrow F = E_0 \oplus E_i$ (as $E_i \not\cong E_0$)

□

Prop 5.2: E : v.b. $\text{rk } E = r$, $\deg E = d$. Then $\forall m \gg 0$,

a gen'l v.b. V w/ $\text{rk } V = mr$,

$$\deg V = mr(g-1) - md,$$

satisfies $H^0(V \otimes E) = H^1(V \otimes E) = 0$.

In particular, L_V is semiample on $M_X^{ss}(r, L)$

Proof: Apply Lemma 5.8 w/ $\mathcal{E} = 0$

$$\Rightarrow H^0(V \otimes E) = 0 \quad \text{for gen'l } V$$

$$\Rightarrow H^1(V \otimes E) = 0$$

Now $E \in M_X^{ss}(r, L)$ ps. $E = \bigoplus_{i=1}^s E_i$

$$\exists V \text{ s.t. } H^i(E_i \otimes V) = 0 \quad \forall i, \forall j=0,1$$

$$\Rightarrow H^i(E \otimes V) = 0$$



$s_V \in T(M_X(r, d), L_V)$ is non-zero at $[E]$

Prop 5.4: E, E' ps. of $rk=r$. $\deg=d$ s.t. E has a direct summand, not a direct summand of E' .

Then $\exists m > 0$ & stable V $\begin{matrix} rk = mr, \\ \deg = mr(g-1) - md, \end{matrix}$

s.t. $H^0(V \otimes E) \neq 0, H^0(V \otimes E') = 0$.

Cor: $s_V \in T(M_{\chi}^{ss}(r, L), L_V)$ separate pts determined by ps bundles for which at least one stable summand is different.

Proof: $E_1, \dots, E_n$ $\xrightarrow{rk=k_i}$ stable summands of E'

E_0 : stable summand of $E \nrightarrow E_1, \dots, E_n$

Goal: find F s.t. $H^0(F \otimes E_0) \neq 0, H^0(F \otimes E_i) = 0$
 $\forall i \geq 1$

Apply lemma 5.9: V gen'l stable verifying

conditions in lemma 5.9. Fix $\phi: E_0^V \xrightarrow{\text{non-zero}} V$.

Want: find a subsheaf $F \subset V$, $\deg F = \deg V - 1$

s.t. ϕ factors F & $\neq$ non-zero $E_i^\vee \rightarrow V$

factoring through F . $\forall 1 \leq i \leq n$.

$$E_0^\vee \rightarrow V \rightarrow Q \rightarrow 0$$

$Q = \text{coker}(\phi)$ is loc. free as $Q = (\text{Ker}(V^\vee \rightarrow E_0))^\vee$

Moreover, $\forall$ non-zero $\psi_i: E_i^\vee \rightarrow V$ $i > 0$

$\leadsto E_0^\vee \oplus E_i^\vee \rightarrow V$ is dual to a surjection Lemma 5.9 iii)

$\Rightarrow E_0^\vee \oplus E_i^\vee \hookrightarrow V \rightarrow V/(E_0^\vee \oplus E_i^\vee)$ loc. free

$\leadsto E_0^\vee \hookrightarrow E_0^\vee \oplus E_i^\vee \hookrightarrow V$ induces

Exact sequence $0 \rightarrow E_i^\vee \xrightarrow{\bar{\psi}_i} Q \rightarrow V/(E_0^\vee \oplus E_i^\vee) \rightarrow 0$ loc. free

$\forall \alpha \in X$, $\exists$ morphism $\text{PHom}(E_i^\vee, V) \xrightarrow{\alpha_i} \text{Gr}(k_i, Q(m))$

$$\psi_i \mapsto \text{Im}(E_i^\vee \rightarrow Q \rightarrow Q(m))$$

$M_i := \text{Im}(\alpha_i)$ has $\dim \leq k_i - 1$ by Lemma 5.9 i)

Claim: $\exists H \subset Q(m)$ hyperplane s.t. $M_i \cap Z_{H,i} = \emptyset$, $\forall i \geq 1$

where $Z_{n,i} \subset Gr(k_i, Q^{(n)})$ consisting of subspaces contained in H .

$$(\dim Z_{n,i} = k_i, \quad Z_{n,i} \cap M_i = \emptyset \text{ for gen'l } H)$$

Fix an $H, \check{E}_i$

$$F := \text{Ker} \left(V \xrightarrow{\quad} V^{(n)} \xrightarrow{\quad} Q^{(n)} \xrightarrow{\quad} Q^{(n)}/H \right)$$

$\text{rk } F = \text{rk } V$
 $\text{deg } F = \text{deg } V - 1$

$$\Rightarrow \phi: E_0^\vee \rightarrow V \text{ factors through } F$$

since $E_0^\vee \rightarrow Q$ is the zero map.

But no non-zero map $E_i^\vee \rightarrow V$ factors through F

since $E_i^\vee \rightarrow Q^{(n)}/H$ is non-zero.

$$\Rightarrow \text{Hom}(E_0^\vee, F) \neq 0, \quad \text{Hom}(E_i^\vee, F) = 0, \quad \forall i \geq 1$$

□

Proof of Theorem:

$\mathcal{L}_V$ is semiample by Prop. 5.2, so

$$\leadsto \phi: M_X^{\text{ss}}(\text{r.d.}) \xrightarrow{|\mathcal{L}_V^{\otimes k}|} \mathbb{P}^N \quad \text{for some } N$$

$$\phi \text{ is } \text{g-} \text{finite} : \forall \text{ ps } E = E_1^{\otimes r_1} \oplus \dots \oplus E_s^{\otimes r_s},$$

$\exists$ finitely many ps v.b. w/ the same stable summands

$$M_X^{\text{ss}} \text{ proper} + \phi \text{ g-finite} \Rightarrow \phi \text{ is finite}$$

In particular, ϕ is affine (hence representable)

$$\Rightarrow M_X^{\text{ss}} \text{ is a scheme, } \phi^*(\mathcal{O}_{\mathbb{P}^N(1)}) \text{ ample.}$$

(Stein factorization)

□

X sep. & g-finite / scheme

$$\Rightarrow X \text{ scheme} \quad (\text{ref. } \underline{\phi})$$

